Second-price auction with two bidders

Consider a second-price auction with two bidders, and suppose that

$$v_1 > v_2$$

1. Find the payoff functions of both players as functions of the bids submitted by each of them
2. Show that, even when ties are broken by a random draw, Nash equilibria exist. You may assign numerical values to the valuations
3. Find, for each player, a weakly dominant strategy. Justify your answer rigorously

Solution

1. Let b_1 and b_2 denote the bids of players 1 and 2

   In a second-price auction, the bidder with the highest bid wins the object and pays the other bidder's bid

   Since this part asks for the payoff functions as functions of the bids, it is convenient to write them case by case, taking into account that if $b_1 = b_2$, the object is assigned by a random draw with probability $\frac{1}{2}$ to each bidder

   For player 1, the payoff function is

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ \frac{1}{2}(v_1 - b_2) & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases}$$

   Indeed, if $b_1 > b_2$, player 1 wins and pays b_2, so his payoff is $v_1 - b_2$

   If $b_1 < b_2$, player 1 loses and gets payoff 0

   If $b_1 = b_2$, player 1 wins with probability $\frac{1}{2}$, pays b_2 if he wins, and otherwise gets 0, so the expected payoff is

$$\frac{1}{2}(v_1 - b_2) + \frac{1}{2}(0) = \frac{1}{2}(v_1 - b_2)$$

   Similarly, for player 2, the payoff function is

$$u_2(b_1, b_2) = \begin{cases} 0 & \text{if } b_1 > b_2 \\ \frac{1}{2}(v_2 - b_1) & \text{if } b_1 = b_2 \\ v_2 - b_1 & \text{if } b_1 < b_2 \end{cases}$$

   Indeed, if $b_2 > b_1$, player 2 wins and pays b_1, so his payoff is $v_2 - b_1$

   If $b_2 < b_1$, player 2 loses and gets payoff 0

   If $b_1 = b_2$, player 2 wins with probability $\frac{1}{2}$, pays b_1 if he wins, and otherwise gets 0, so the expected payoff is

$$\frac{1}{2}(v_2 - b_1) + \frac{1}{2}(0) = \frac{1}{2}(v_2 - b_1)$$

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ \frac{1}{2}(v_1 - b_2) & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases} \qquad u_2(b_1, b_2) = \begin{cases} 0 & \text{if } b_1 > b_2 \\ \frac{1}{2}(v_2 - b_1) & \text{if } b_1 = b_2 \\ v_2 - b_1 & \text{if } b_1 < b_2 \end{cases}$$

2. A simple candidate is the bid profile

$$b_1 = v_1 \qquad b_2 = v_2$$

   Since $v_1 > v_2$, player 1 wins the auction and pays the second price, which is v_2

   Thus, the payoffs are

$$u_1 = v_1 - v_2 \qquad u_2 = 0$$

   We now check that neither player has a profitable unilateral deviation

   Player 1

If player 1 deviates to some bid $b'_1 > v_2$, she still wins and still pays v_2, so her payoff remains

$$v_1 - v_2$$

If player 1 deviates to

$$b'_1 = v_2$$

then there is a tie, and since ties are broken randomly, player 1 wins with probability $\frac{1}{2}$
Her expected payoff becomes

$$\frac{1}{2}(v_1 - v_2)$$

which is strictly smaller than

$$v_1 - v_2$$

If player 1 deviates to some bid $b'_1 < v_2$, she loses and gets

$$0$$

which is also strictly smaller than

$$v_1 - v_2$$

Therefore, player 1 has no profitable deviation

Player 2
If player 2 deviates to some bid $b'_2 < v_1$, she still loses and gets

$$0$$

If player 2 deviates to

$$b'_2 = v_1$$

then there is a tie, so her expected payoff is

$$\frac{1}{2}(v_2 - v_1)$$

Since $v_1 > v_2$, this is negative
If player 2 deviates to some bid $b'_2 > v_1$, she wins but pays v_1, so her payoff is

$$v_2 - v_1$$

which is also negative
Thus, player 2 also has no profitable deviation
Hence,

$$(b_1, b_2) = (v_1, v_2) \text{ is a Nash equilibrium}$$

To make this fully explicit with numbers, take for example

$$v_1 = 10 \qquad v_2 = 6$$

Then

$$(b_1, b_2) = (10, 6)$$

is a Nash equilibrium. Player 1 gets payoff 4, player 2 gets payoff 0, and no unilateral deviation improves either player's payoff

Therefore, Nash equilibria do exist even when ties are broken by a random draw

3. We now show that, for each player i, bidding her true valuation

$$b_i = v_i$$

is a weakly dominant strategy

Recall that a strategy s_i is weakly dominant if, for every possible bid of the opponent, it gives a payoff at least as high as any other strategy, and for some cases it gives a strictly higher payoff

We prove this separately for a generic player i, with valuation v_i, facing an opponent bid b_{-i}

Case 1: $b_{-i} < v_i$

If player i bids

$$b_i = v_i$$

then she wins, pays b_{-i}, and obtains payoff

$$v_i - b_{-i} > 0$$

Now compare this with any alternative bid b_i'

If

$$b_i' > b_{-i}$$

then player i still wins and still pays b_{-i}, so the payoff is again

$$v_i - b_{-i}$$

If

$$b_i' = b_{-i}$$

then there is a tie, so the expected payoff is

$$\frac{1}{2}(v_i - b_{-i})$$

which is smaller than

$$v_i - b_{-i}$$

If

$$b'_i < b_{-i}$$

then player i loses and gets

$$0$$

which is also smaller than

$$v_i - b_{-i}$$

Thus, when

$$b_{-i} < v_i$$

bidding v_i yields a payoff at least as high as any other bid

Case 2: $b_{-i} = v_i$

If player i bids

$$b_i = v_i$$

then there is a tie, so the expected payoff is

$$\frac{1}{2}(v_i - v_i) = 0$$

Now consider any alternative bid b'_i

If

$$b'_i > v_i$$

then player i wins for sure, pays v_i, and gets

$$v_i - v_i = 0$$

If

$$b'_i = v_i$$

the payoff is again

$$0$$

If

$$b'_i < v_i$$

then player i loses and gets

$$0$$

So in this case every bid gives payoff 0, and truthful bidding is therefore at least as good as any other bid

Case 3: $b_{-i} > v_i$
If player i bids

$$b_i = v_i$$

then she loses and gets

$$0$$

Now compare this with any alternative bid b_i'
If

$$b_i' < b_{-i}$$

then player i still loses and gets

$$0$$

If

$$b_i' = b_{-i}$$

then there is a tie, so the expected payoff is

$$\frac{1}{2}(v_i - b_{-i}) < 0$$

If

$$b_i' > b_{-i}$$

then player i wins and pays b_{-i}, so the payoff is

$$v_i - b_{-i} < 0$$

Hence, when

$$b_{-i} > v_i$$

bidding truthfully gives payoff 0, while any bid that makes the player win yields a negative payoff
Therefore, truthful bidding is again at least as good as any alternative

Since these three cases exhaust all possibilities for the opponent's bid, we conclude that for any player i,

$$b_i = v_i \text{ is a weakly dominant strategy}$$

Applying this to the two players in the exercise, we obtain

$$b_1 = v_1 \qquad b_2 = v_2$$

Therefore, in a second-price auction, each player's weakly dominant strategy is to bid her true valuation